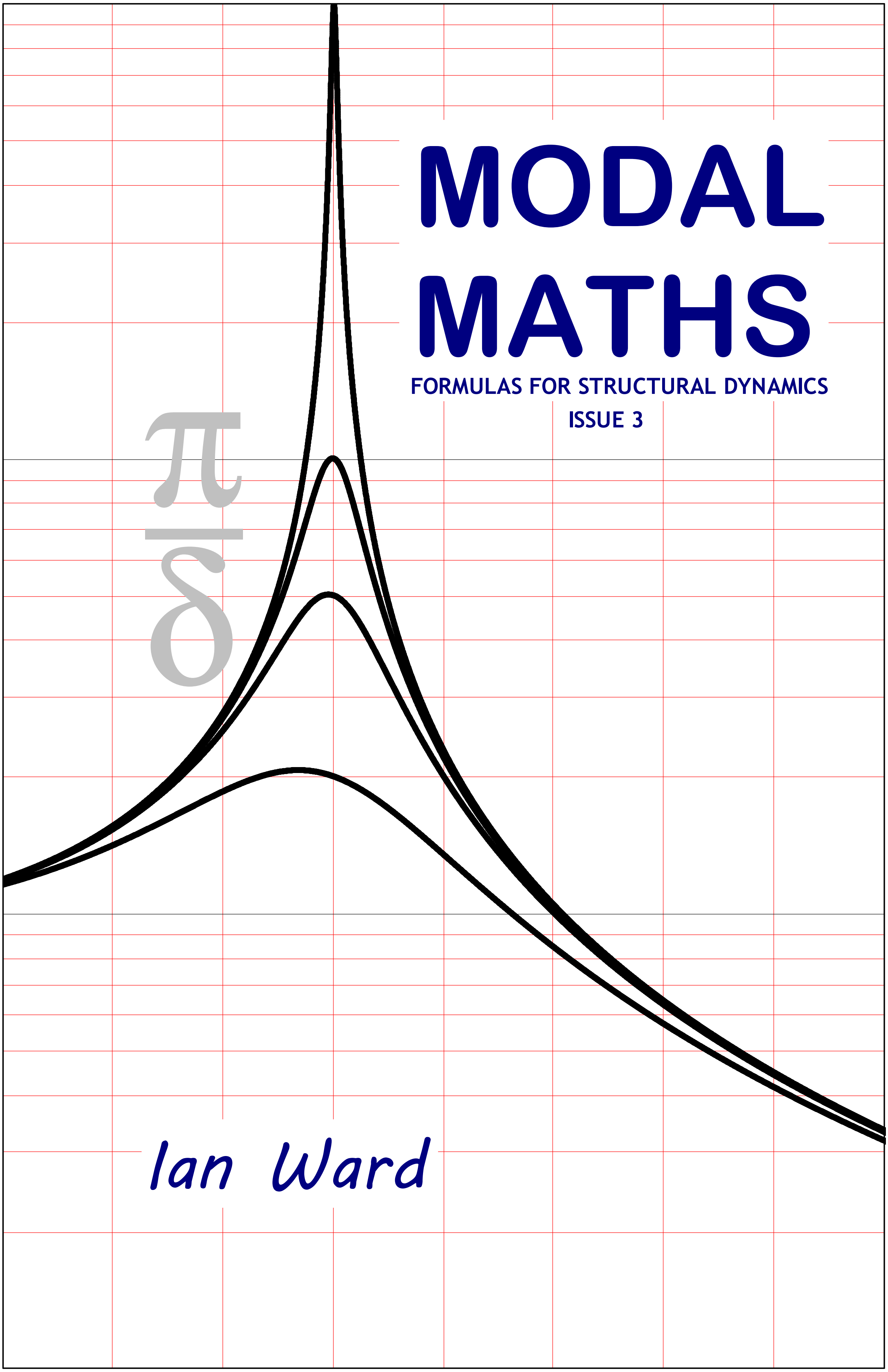
MODAL MATHS

FORMULAS FOR STRUCTURAL DYNAMICS

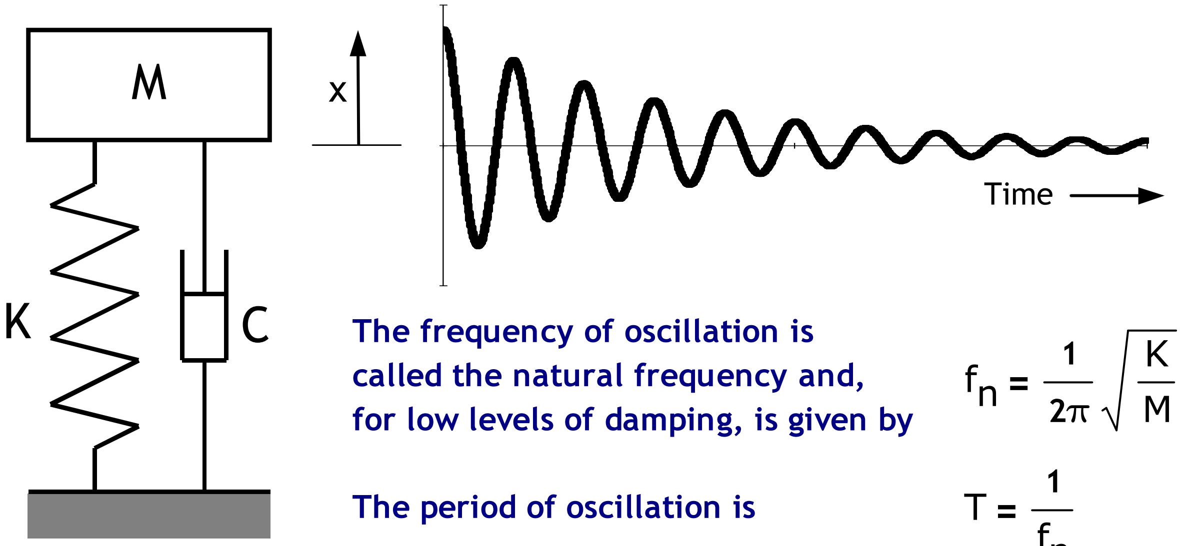
ISSUE 3



Ian Ward

Natural frequency

The single degree of freedom, or SDOF, system is a useful concept in structural dynamics. It consists of a mass M connected to ground via a spring with stiffness K and a damper with damping coefficient C. If the mass is given an initial displacement (x) and then released, the displacement oscillates about zero and gradually decreases with time.



The frequency of oscillation is called the natural frequency and, for low levels of damping, is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$$

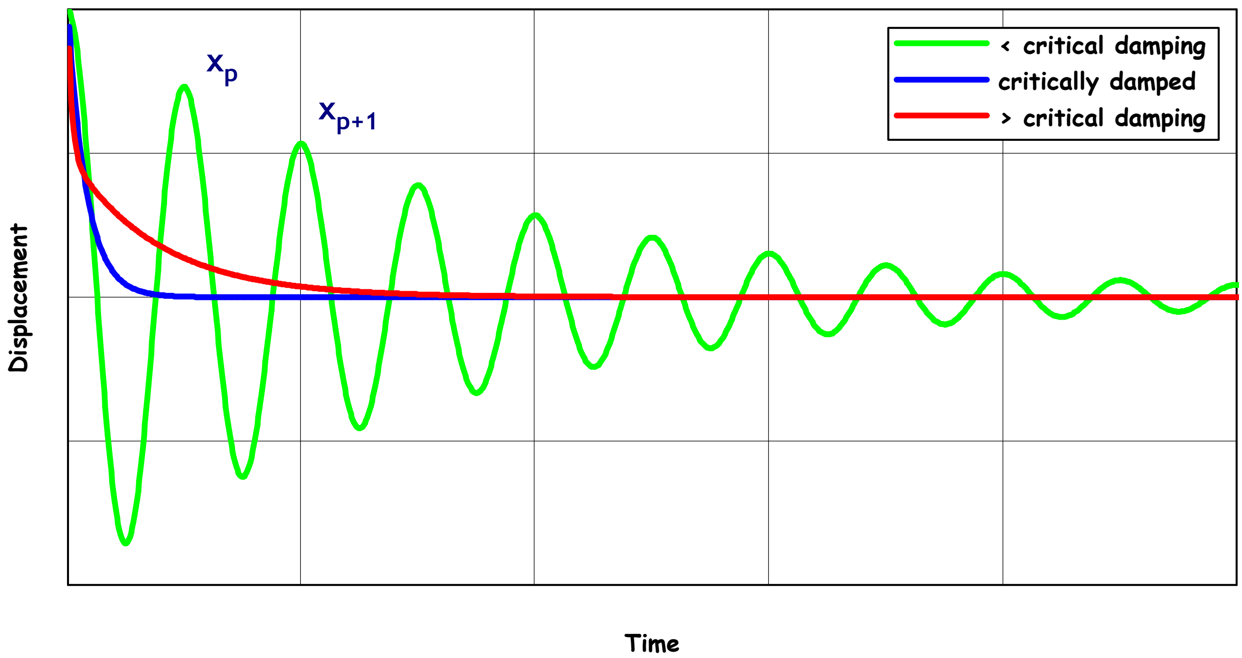
The period of oscillation is

$$T = \frac{1}{f_n}$$

The rate of decay of the response is governed by damping. For structural applications this is expressed as a 'damping ratio' or as a 'logarithmic decrement'.

Damping Ratio

If the damping were to be gradually increased until the displacement dropped to zero without any oscillation then the SDOF would be said to be 'critically damped' with a damping coefficient of C_{crit} (indicated by the blue curve in the graph below).



Any further increase in the damping would give rise to a slower decay to zero (the red curve in the graph). In practice, structural damping is only a fraction of the critical value and the decay of vibration is more closely represented by the green curve in the graph. The amount of damping can be defined in terms of a critical damping ratio:

damping ratio $$\xi = \frac{C}{C_{crit}}$$

The relationship between the damping ratio and the damping coefficient is

$$C = 2\xi M\omega = 2\xi\sqrt{MK}$$

with the circular frequency ω given by $\omega = 2\pi f$

Logarithmic Decrement

An alternative way of describing the structural damping is to consider the height of successive peaks in the vibration decay (denoted as x_p and x_{p+1} in the graph on the previous page). The natural logarithm of this ratio is the logarithmic decrement δ (or 'log dec'):

logarithmic decrement $$\delta = \ln\left(\frac{x_p}{x_{p+1}}\right) \quad \text{i.e.} \quad \frac{x_p}{x_{p+1}} = e^{\delta}$$

Or, if the decay over a number of cycles N is considered then

logarithmic decrement $$\delta = \frac{1}{N}\ln\left(\frac{x_p}{x_{p+N}}\right) \quad \text{i.e.} \quad \frac{x_p}{x_{p+N}} = e^{N\delta}$$

The relationship between the logarithmic decrement and the damping ratio is... $\delta = 2\pi\xi$

As a rough guide, a logarithmic decrement of 0.1 means that the peak amplitude falls by approximately 10% in each successive cycle.

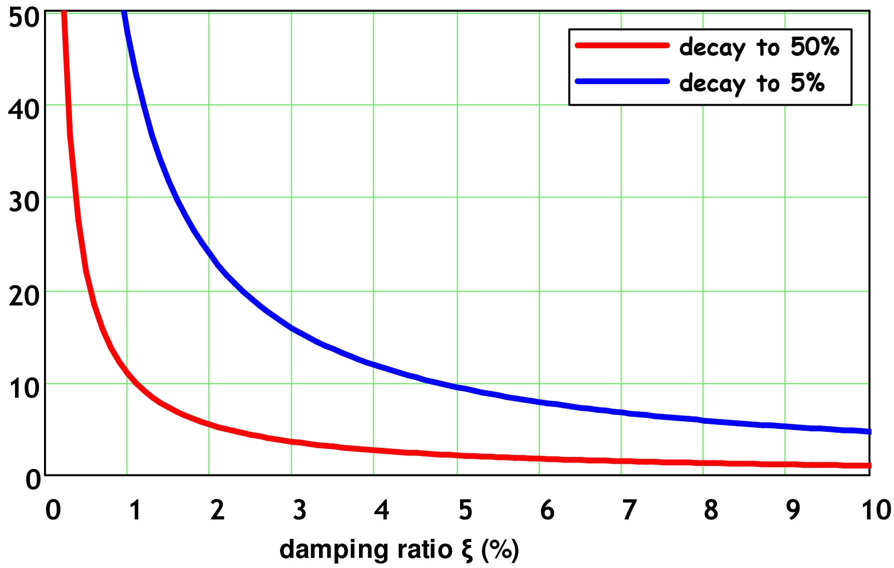
Another way of visualizing the vibration decay associated with a particular damping value is to consider the number of cycles required to cause the peak amplitude to decay to a certain level.

decay to 50% ... $$N_{50} = \frac{\ln\left(\frac{1}{0.5}\right)}{\delta} = \frac{\ln(2)}{\delta} = \frac{\ln(2)}{2\pi\xi}$$

decay to 5% ... $$N_5 = \frac{\ln\left(\frac{1}{0.05}\right)}{\delta} = \frac{\ln(20)}{2\pi\xi} = \frac{1}{2\xi} = \frac{\pi}{\delta} \quad \text{(approx)}$$

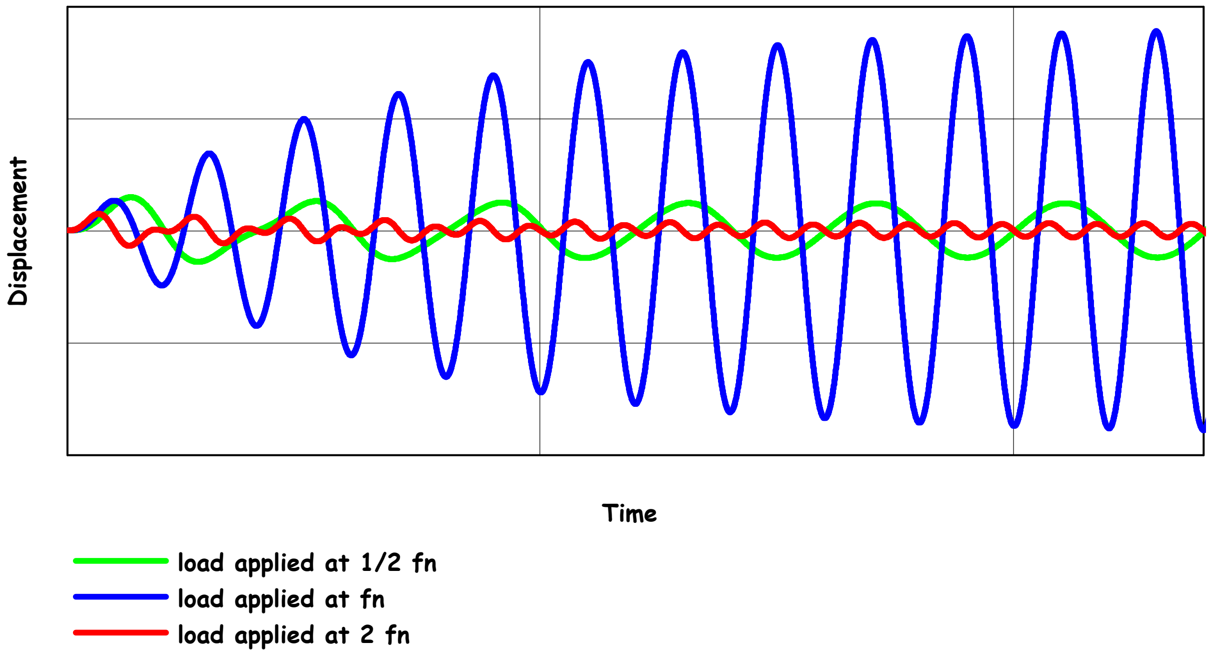
(these relationships are shown in the graph overleaf)

Number of cycles, N, required for amplitude to decay to a given percentage of the original level.



Dynamic Magnification Factor

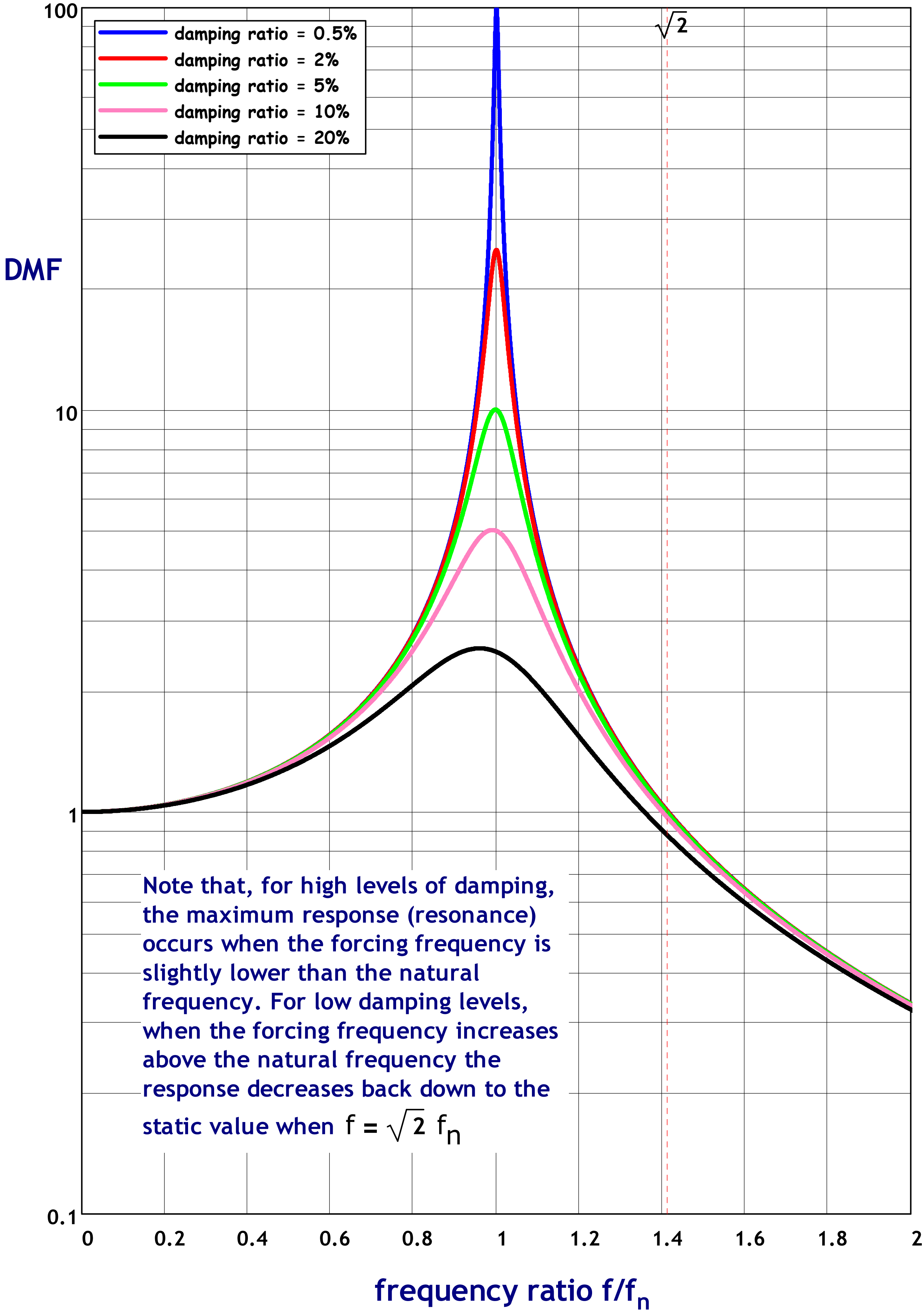
If a sinusoidal load is applied to the SDOF the response eventually reaches a 'steady-state' condition. The graph below shows the response of a SDOF with 5% damping ratio when the sinusoidal load is applied at three different frequencies (but with the same load amplitude).



The peak value of the steady-state deflection, as a proportion of the static deflection, is called the 'Dynamic Magnification Factor', given by:

$$\text{DMF} = \frac{1}{\sqrt{\left[1 - \left(\frac{f}{f_n}\right)^2\right]^2 + \left(2\xi\,\frac{f}{f_n}\right)^2}}$$

DMF versus frequency ratio for different levels of damping



The maximum DMF occurs when the forcing frequency is

$$f_{max} = f_n\sqrt{1 - 2\xi^2}$$

The value of the maximum DMF is

$$DMF_{max} = \frac{1}{2\xi} = \frac{\pi}{\delta}$$

From this it follows that the displacement at resonance is

$$x_{res} = \frac{1}{2\xi}\frac{F}{K} = \frac{\pi}{\delta}\frac{F}{K}$$

and the acceleration at resonance is

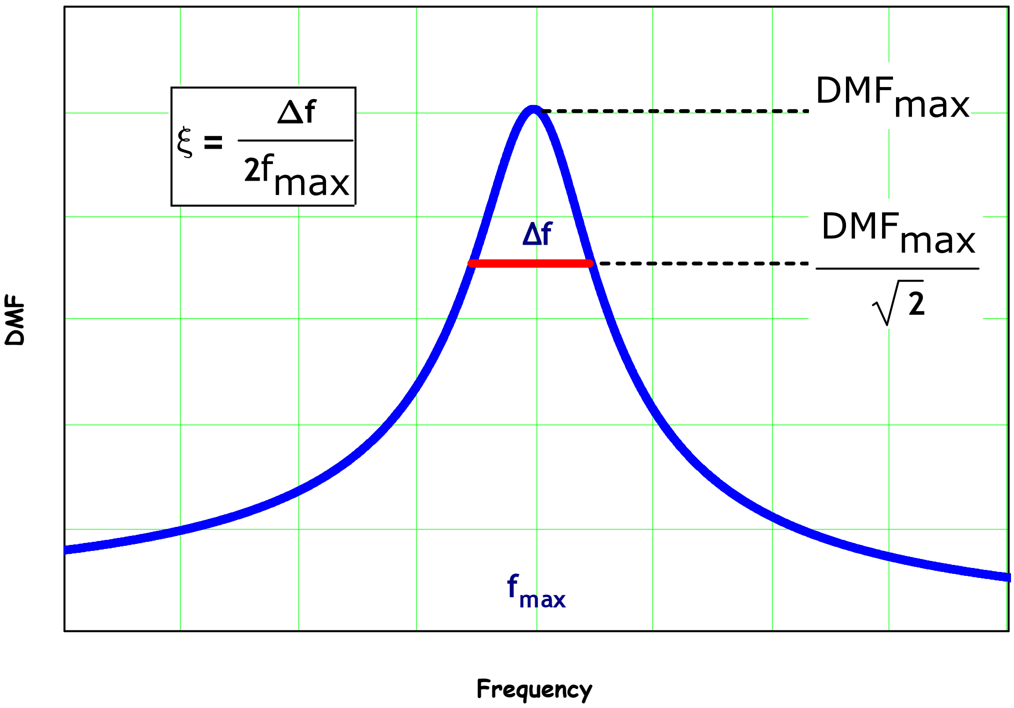
$$a_{res} = \frac{1}{2\xi}\frac{F}{M} = \frac{\pi}{\delta}\frac{F}{M}$$

The DMF curve also supplies a means for determining the damping ratio:

$$\xi = \frac{\Delta f}{2f_{max}}$$

where Δf is the width of the DMF curve at $\frac{1}{\sqrt{2}}$ times the resonant amplitude at frequency f_{max}.

This is referred to as the half-power bandwidth method for determining the damping, and is shown graphically in the figure below.

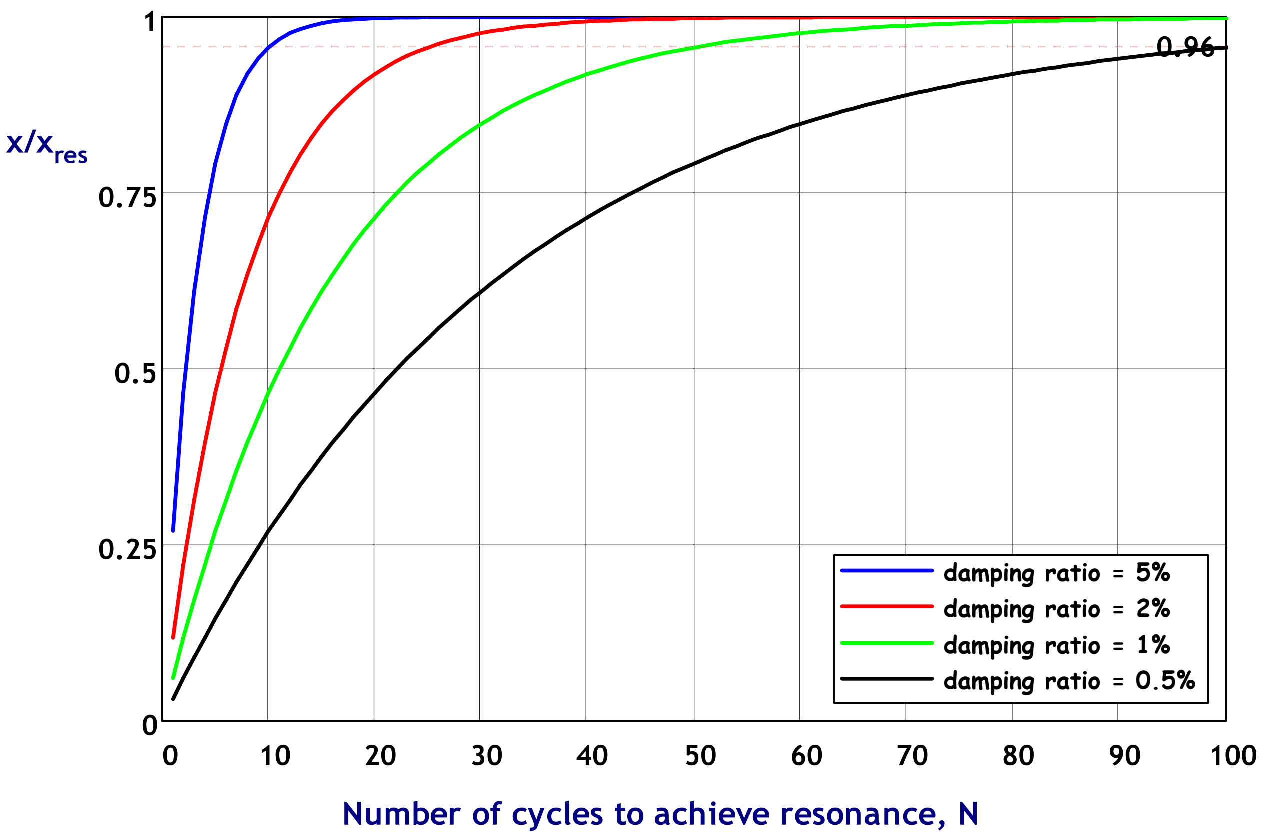


Build-up of Resonant Response

The response after a number of cycles, N, as a proportion of the final resonant response is given by:

$$\frac{x}{x_{res}} = \left(1 - e^{-2\pi\xi N}\right)$$

This expression is illustrated in the figure below for a number of different damping ratios.



An approximate relationship for the number of cycles required to reach the maximum resonant response (strictly speaking, 96% of it) is:

$$N_{peak} = \frac{1}{2\xi} = \frac{\pi}{\delta}$$

The figure shows that roughly the same number of cycles again are required for the response to build up from 96% to 100% of the maximum resonant response.

Formulas for natural frequency

Undamped natural frequency of system with stiffness K and mass M

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{M}}$$

Damped natural frequency

$$f_d = f_n\sqrt{1-\xi^2}$$

(This shows that the damped natural frequency of a structure with 5% damping will only be 0.1% lower than the undamped natural frequency. This means that, for typical engineering structures, it can be assumed that $f_d = f_n$).

Natural frequency of a SDOF system in terms of the self-weight deflection Δ caused by 1g (Blevins (1979)).

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

i.e. $$f_n = \frac{15.76}{\sqrt{\Delta}}$$ (with Δ in mm)

Modified version of the equation above. Approximate value for the natural frequency of a structure with distributed mass and stiffness (Steel Designer's manual (2003)).

$$f_n = \frac{18}{\sqrt{\Delta}}$$ (with Δ in mm)

String/cable of length L with tension T and mass per unit length m.

$$f_n = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Pendulum of length L.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Undamped natural frequency of system with torsional stiffness K (moment/rotation) and mass moment of inertia J.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{J}}$$

Approximate formula for road bridge of length L metres (Bachmann & Ammann (1995)).

$$f_n = \frac{100}{L}$$

Approximate formula for building of height H metres (Ellis (1980)).

$$f_n = \frac{46}{H}$$

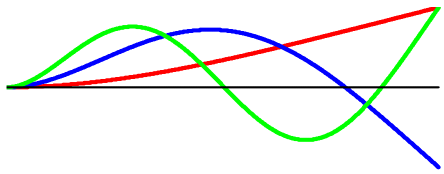
Modal properties of uniform beams with various support conditions (Blevins (1979))

Natural frequency of mode i

$$f_i = \frac{\lambda_i^2}{2\pi L^2}\sqrt{\frac{EI}{m}}$$

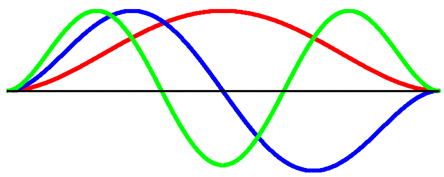
E=Young's modulus, L=length, I=area moment of inertia, m=mass per unit length. (see below for values of λ)

Fixed - Free beam



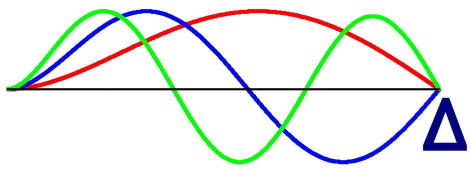
Mode	λ	Generalized mass (fraction of total mass)	Participation factor
i = 1	1.875	0.250	1.566
i = 2	4.694	0.250	0.867
i = 3	7.855	0.250	0.509

Fixed - Fixed beam



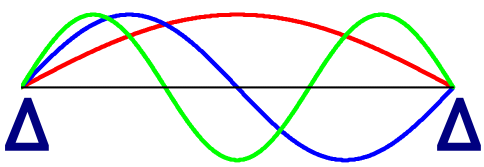
Mode	λ	Generalized mass (fraction of total mass)	Participation factor
i = 1	4.730	0.396	1.320
i = 2	7.853	0.439	0.000
i = 3	10.996	0.437	0.550

Fixed - Pinned beam



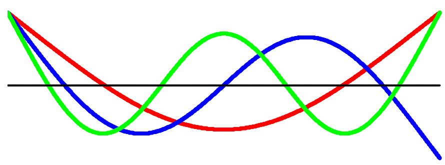
Mode	λ	Generalized mass (fraction of total mass)	Participation factor
i = 1	3.927	0.439	1.298
i = 2	7.069	0.437	0.125
i = 3	10.210	0.437	0.506

Pinned - Pinned beam



Mode	λ	Generalized mass (fraction of total mass)	Participation factor
i = 1	π	0.500	1.273
i = 2	2π	0.500	0.000
i = 3	3π	0.500	0.424

Free - Free beam



Mode	λ	Generalized mass (fraction of total mass)	Participation factor
i = 1	4.730	0.250	0.000
i = 2	7.853	0.250	0.000
i = 3	10.996	0.250	0.000

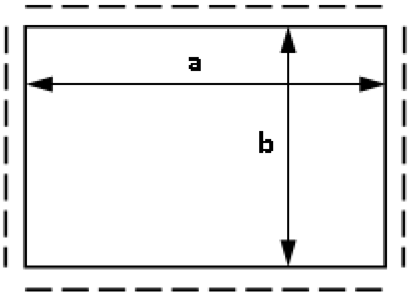
Fundamental frequencies of plates with various support conditions

(Bachmann & Ammann (1995))

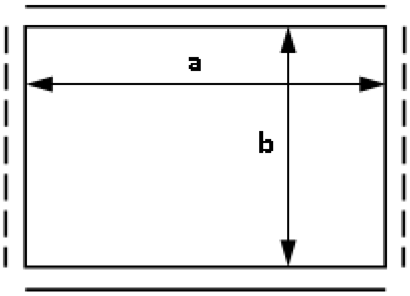
Natural frequency

$$f = \frac{\lambda}{a^2}\sqrt{\frac{Et^2}{12\ \rho\left(1-\nu^2\right)}}$$

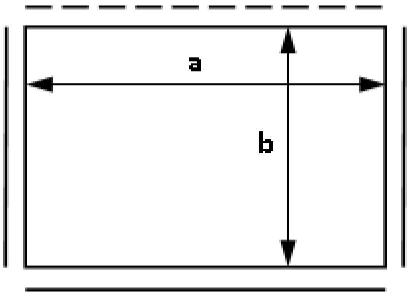
E=Young's modulus, t=thickness, ρ=density, a=length, b=width, μ=a/b, ν=Poisson's ratio (see below for values of λ)



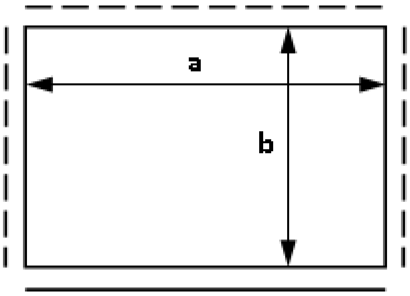
$$\lambda = 1.57\left(1+\mu^2\right)$$



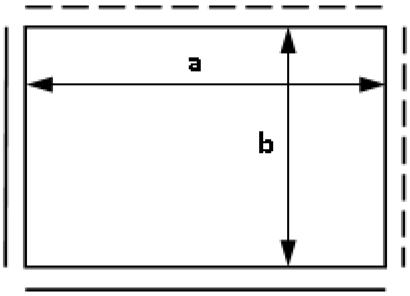
$$\lambda = 1.57\sqrt{\left(1+2.5\ \mu^2+5.14\ \mu^4\right)}$$



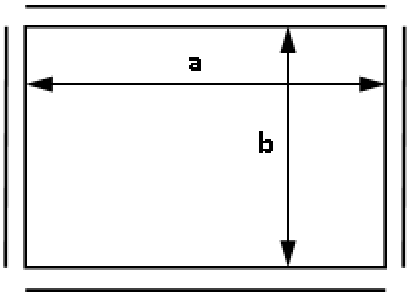
$$\lambda = 1.57\sqrt{\left(5.14+2.92\ \mu^2+2.44\ \mu^4\right)}$$



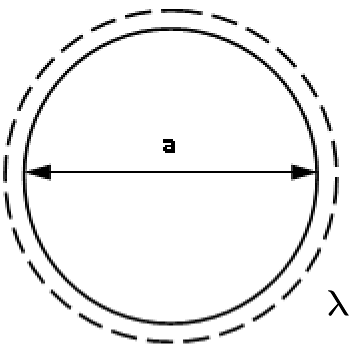
$$\lambda = 1.57\sqrt{\left(1+2.33\ \mu^2+2.44\ \mu^4\right)}$$



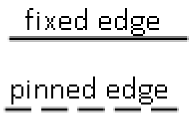
$$\lambda = 1.57\sqrt{\left(2.44+2.72\ \mu^2+2.44\ \mu^4\right)}$$

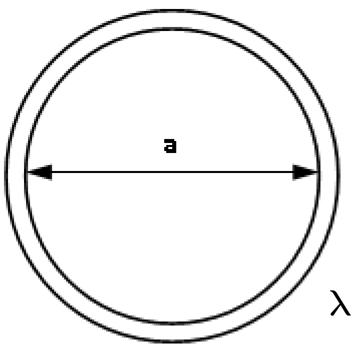


$$\lambda = 1.57\sqrt{\left(5.14+3.13\ \mu^2+5.14\ \mu^4\right)}$$



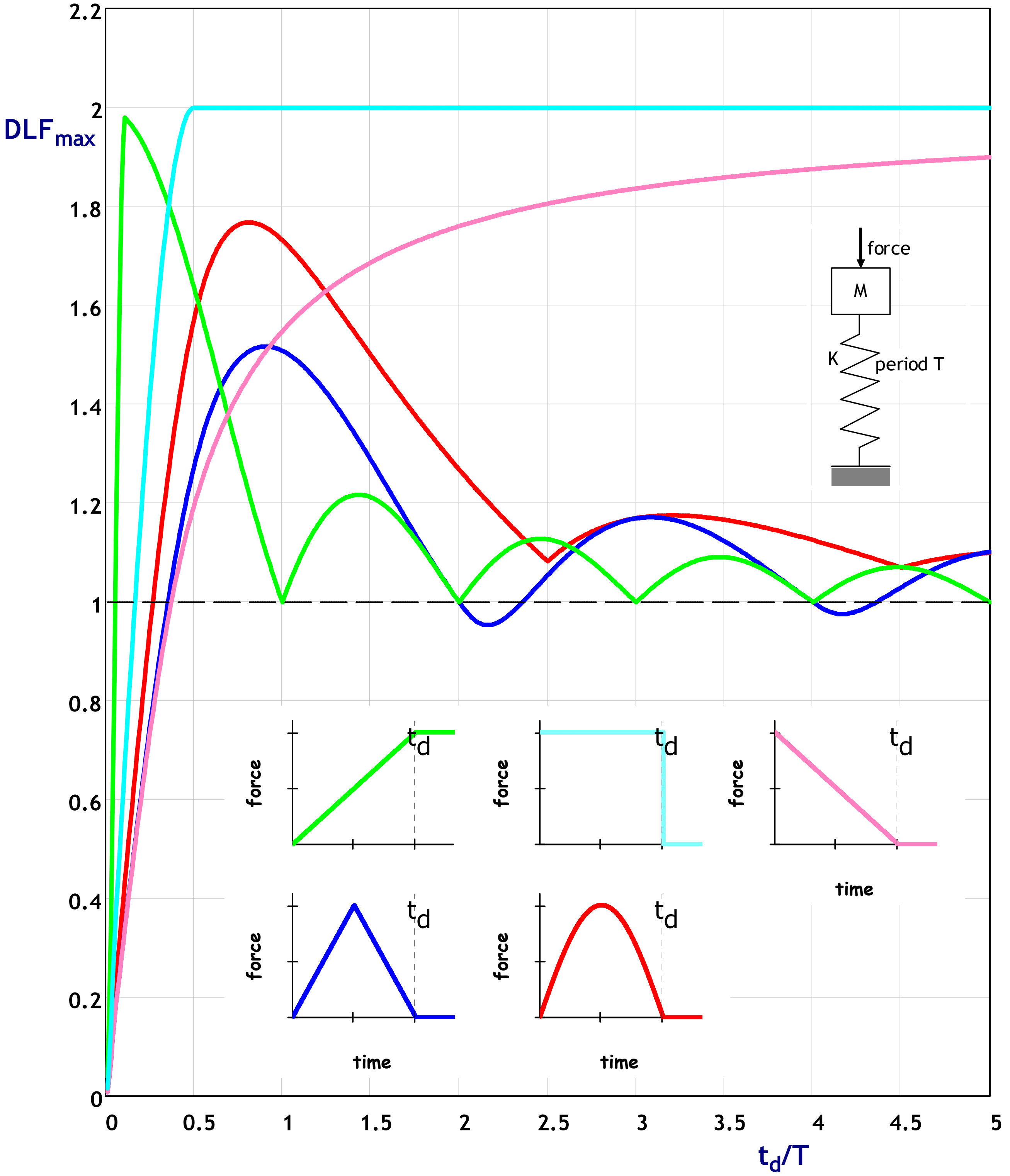
$\lambda = 3.25$





$\lambda = 6.53$

Maximum Response of an Undamped SDOF Elastic System Subject to Various Load Pulses (Biggs (1964))

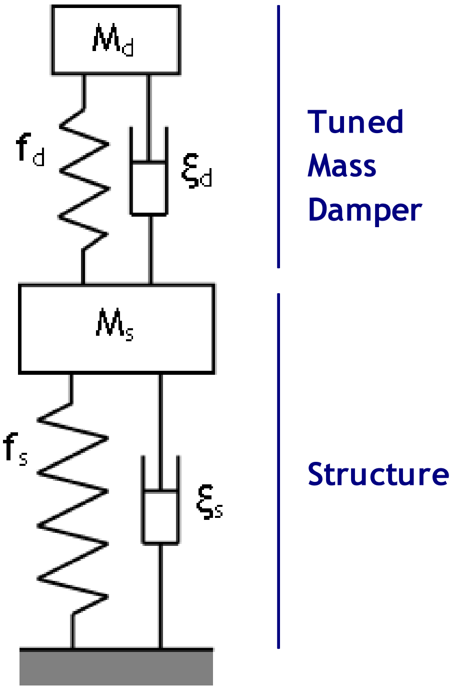


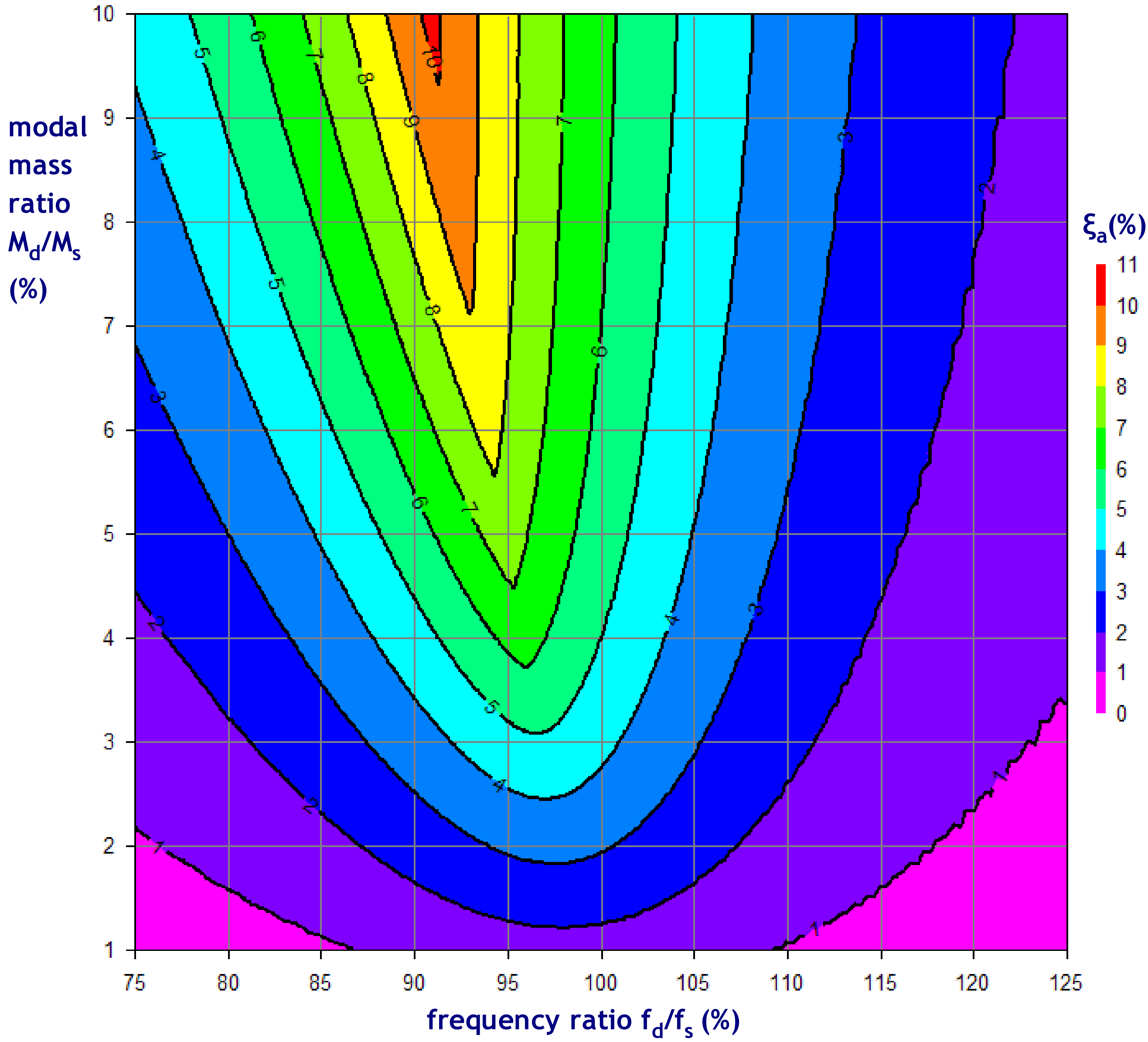
In the derivation of these charts no damping has been included because it has no significant effect. The maximum DLF usually corresponds to the first peak of response, and the amount of damping normally encountered in structures is not sufficient to significantly decrease this value.

Tuned Mass Dampers

The TMD design chart below gives the additional overall damping ratio (ξ_a) provided by a TMD with active mass M_d when attached to a structure with modal mass M_s and inherent damping ξ_s. The chart is based on a TMD with ξ_d = 15% and a structure with ξ_s in the range 0.5% - 2.5%. It also assumes that the TMD is located at the position of maximum response of the mode being damped.

The overall damping becomes $\xi_s + \xi_a$.





Tuned Mass Dampers

If the TMD can be tuned then the TMD chart shows that the additional damping is roughly

$$\xi_a = \frac{M_d}{M_s}$$

Maximum deflection of TMD relative to structure

$$xrel_d = \frac{1}{2\xi_d} = \frac{\pi}{\delta_d}$$

Optimum TMD frequency (Mead (1998))

$$f_d = \frac{f_s}{1+\mu} \quad \text{with} \quad \mu = \frac{M_d}{M_s}$$

Optimum TMD damping ratio (Bachmann & Ammann (1987))

$$\xi_d = \sqrt{\frac{3\,\mu}{8(1+\mu)^3}}$$

Wind-induced vortex shedding

Critical wind speed for vortex shedding (Blevins 2001)

$$v_{crit} = \frac{f_n\,D}{St}$$

f_n=natural frequency, D=across-wind dimension, St=Strouhal number.

For a circular cylinder St is approximately 0.2 and therefore $v_{crit} = 5\,f_n\,D$

The susceptibility of vortex-induced vibrations depends on the structural damping and the ratio of the structural mass to the fluid mass. This is expressed by the Scruton number (Sc), also known as the 'mass-damping parameter' (Scruton 1981).

$$Sc = \frac{2\,m_e\,\delta_s}{\rho_{air}\,D^2} \quad \text{with} \quad m_e = \frac{\int_0^L m(x)\,\phi(x)^2\,dx}{\int_0^L \phi(x)^2\,dx}$$

(equivalent mass per unit length)

Wind-induced galloping den Hartog (1956)

Critical wind speed for galloping

$$v_{crit} = \frac{2\,Sc\,f_n\,D}{\frac{dC_y}{d\alpha}}$$

$dC_y/d\alpha$ is the rate of change of the lateral force coefficient with angle of attack

A section is susceptible to galloping if

$$\frac{dC_y}{d\alpha} > 0$$

note that

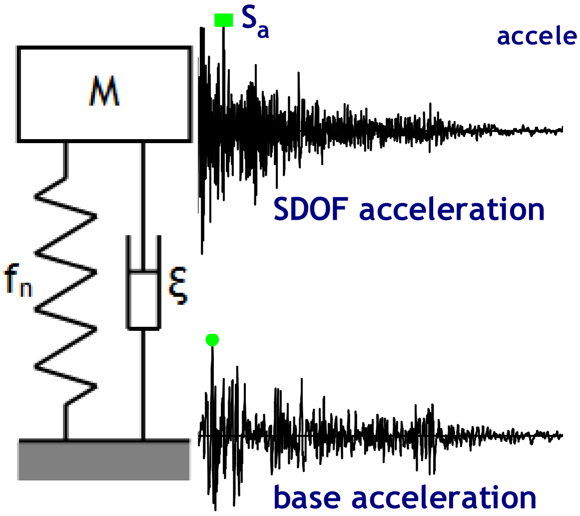
$$\frac{dC_y}{d\alpha} = -\left(\frac{dC_L}{d\alpha} + C_D\right)$$

with C_L=lift coefficient, C_D=drag coefficient

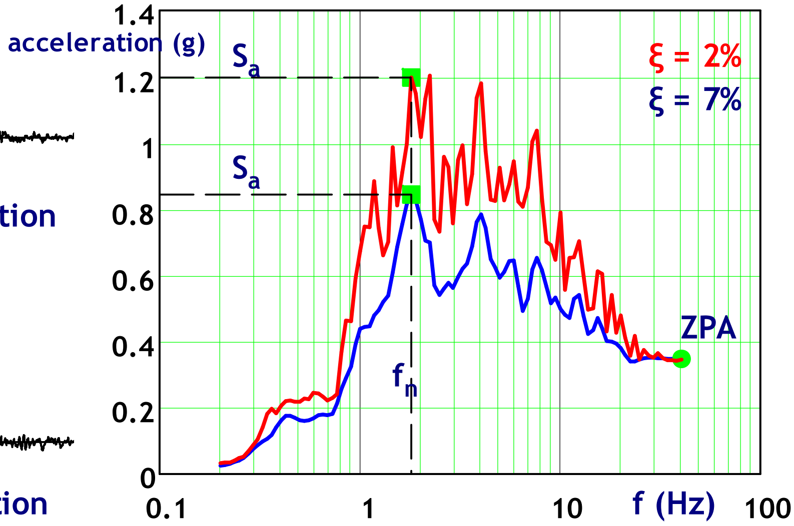
Seismic Response Spectrum Analysis

Relationship between SDOF response and structural response for each mode.

Single Degree of Freedom

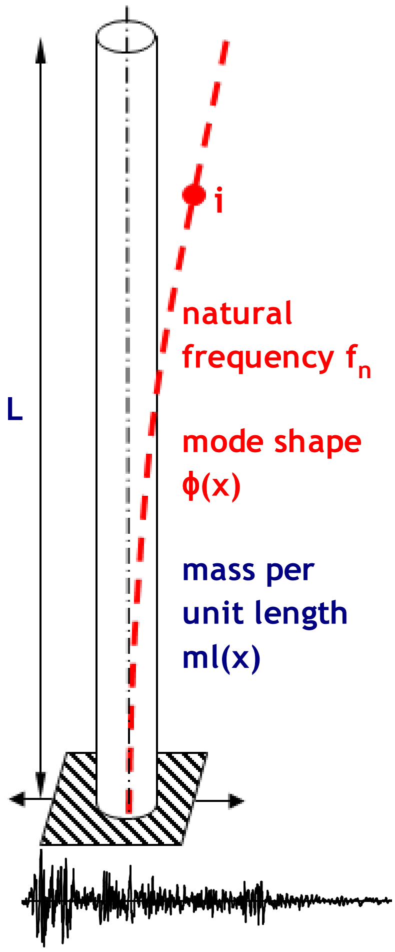


response spectrum



Approximate relationship $$S_a = \frac{ZPA}{\sqrt{2\xi}} = ZPA\sqrt{\frac{\pi}{\delta}}$$

Structure with distributed mass and stiffness



Generalized mass (modal mass) $$M_G = \int_0^L \phi(x)^2\, ml(x)\, dx$$

Participation factor (see page 8 for typical values) $$\Gamma = \frac{\int_0^L \phi(x)\, ml(x)\, dx}{\int_0^L \phi(x)^2\, ml(x)\, dx}$$

Maximum acceleration at point i $$a_i = \Gamma\, \phi(x_i)\, S_a$$

Effective mass $$M_E = \frac{\left(\int_0^L \phi(x)\, ml(x)\, dx\right)^2}{\int_0^L \phi(x)^2\, ml(x)\, dx}$$

Base shear force $$SF_{base} = M_E\, S_a = \Gamma^2\, M_G\, S_a$$

The maximum response from each of the modes can be combined using the 'Square Root Sum of Squares' or 'Complete Quadratic Combination' method (see next page).

Combination of Modal Responses

This section compares two standard techniques for combining the response from different modes, namely the SRSS (Square Root Sum of Squares) and CQC (Complete Quadratic Combination) methods. As the name suggests, the SRSS method calculates the combined response by performing an SRSS on the modal contributions. For the CQC method the modal responses, a, are combined by making use of modal cross-correlation coefficients (Wilson, Kiureghian & Bayo):

$$a\_SRSS = \sqrt{\sum_i (a_i)^2} \qquad a\_CQC = \sqrt{\sum_i \sum_j (a_i\ \rho_{i,j}\ a_j)}$$

where i, j represent the modes and

$$\rho_{i,j} = \frac{8\sqrt{\xi_i\ \xi_j}\left(\xi_i + \frac{f_i}{f_j}\xi_j\right)\left(\frac{f_i}{f_j}\right)^{1.5}}{\left[1-\left(\frac{f_i}{f_j}\right)^2\right]^2 + 4\ \xi_i\ \xi_j\ \frac{f_i}{f_j}\left[1+\left(\frac{f_i}{f_j}\right)^2\right] + 4\left[(\xi_i)^2+(\xi_j)^2\right]\left(\frac{f_i}{f_j}\right)^2}$$

where f=frequency and ξ=damping ratio. The difference between the two methods is illustrated with some examples based on 4 modes, each with a=1:

$$f = \begin{pmatrix} 3 \\ 5 \\ 7 \\ 9 \end{pmatrix} \text{Hz} \qquad \xi = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \% \qquad \rho = \begin{pmatrix} 1 & 0.01 & 0 & 0 \\ 0.01 & 1 & 0.01 & 0 \\ 0 & 0.01 & 1 & 0.02 \\ 0 & 0 & 0.02 & 1 \end{pmatrix}$$

Well-spaced modes
a_SRSS = 2
a_CQC = 2.03

$$f = \begin{pmatrix} 3 \\ 5 \\ 5.01 \\ 9 \end{pmatrix} \text{Hz} \qquad \xi = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \% \qquad \rho = \begin{pmatrix} 1 & 0.01 & 0.01 & 0 \\ 0.01 & 1 & 1 & 0 \\ 0.01 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Two of the modes now closely spaced
a_SRSS = 2
a_CQC = 2.46

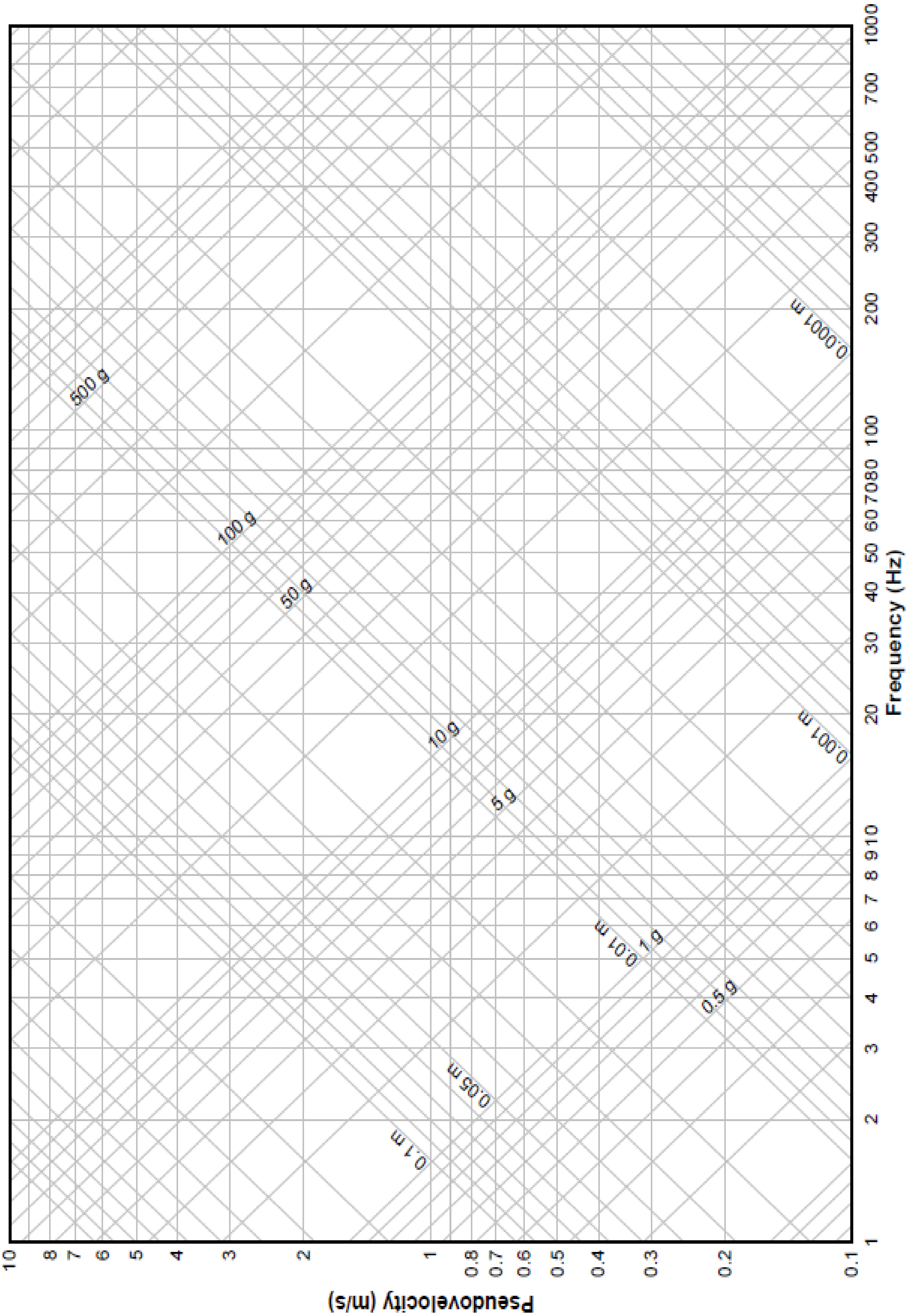
$$f = \begin{pmatrix} 3 \\ 5 \\ 7 \\ 9 \end{pmatrix} \text{Hz} \qquad \xi = \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix} \% \qquad \rho = \begin{pmatrix} 1 & 0.13 & 0.05 & 0.03 \\ 0.13 & 1 & 0.26 & 0.1 \\ 0.05 & 0.26 & 1 & 0.38 \\ 0.03 & 0.1 & 0.38 & 1 \end{pmatrix}$$

Well-spaced modes with increased ξ
a_SRSS = 2
a_CQC = 2.42

$$f = \begin{pmatrix} 5 \\ 5.01 \end{pmatrix} \text{Hz} \qquad \xi = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \% \qquad \rho = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Two close modes
a_SRSS = 1.414
a_CQC = 2

Tripartite Graph Paper



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Terminology

δ logarithmic decrement

Δf frequency interval

ξ damping ratio

ω circular frequency ($2\pi f$)

v Poisson's ratio

Γ participation factor

ϕ mode shape

μ aspect ratio, mass ratio

ρ density

Δ self-weight deflection

a acceleration, amplitude

C damping coefficient

C_{crit} critical damping coefficient

C_D drag coefficient

C_L lift coefficient

C_y lateral force coefficient

CQC complete quadratic combination

D across-wind dimension

DMF dynamic magnification factor

E Young's modulus

f frequency

f_d damped natural frequency

f_i natural frequency of mode i

f_n undamped natural frequency

g acceleration due to gravity

h plate thickness

H height

J mass moment of inertia

K stiffness

L length

ml mass per unit length

M mass

M_E effective mass

M_G generalized mass

N number of cycles

SDOF single degree of freedom

Sc Scruton number

St Strouhal number

SRSS square root sum of squares

t time, thickness

T period, tension

x displacement

ZPA zero period acceleration

INDEX