Appendix A

THE RELATIONSHIP BETWEEN STATIC DEFLECTION AND FUNDAMENTAL NATURAL FREQUENCY

General Case. As noted in Chapter 4, the natural frequency of a structure is the result of the exchange of kinetic and potential energy within the structure. The kinetic energy is associated with the motion of structural mass, and the potential energy is associated with the strain energy stored in the elastic structure during deformation. One measure of the strain energy stored in a structure is the static deflection of the structure under the acceleration of gravity. Thus, it is reasonable to believe that a relationship exists between the static deflection of a structure under its own weight and the natural frequency of the structure. This relationship can be used to estimate the fundamental natural frequency of complex structures whose static deflection is known.

For example, the static deflection of the mass in the spring-mass system shown in Fig. A-1 (a) due to gravity is (Eq. 4-9):

$$\delta_s = \frac{Mg}{k}, \tag{A-1}$$

where M is the mass, g is the acceleration due to gravity, and k is the spring constant. The natural frequency of this system is (Eq. 4-6 or frame 1 of Table 6-2):

$$f = \frac{1}{2\pi}\left(\frac{k}{M}\right)^{1/2} \tag{A-2}$$

Incorporating Eq. A-1 into Eq. A-2 to eliminate the spring constant k gives:

$$f = \frac{1}{2\pi}\left(\frac{Mg}{\delta_s M}\right)^{1/2}$$

or

$$f = \frac{1}{2\pi}\left(\frac{g}{\delta_s}\right)^{1/2} \text{ Hz}. \tag{A-3}$$

Equation A-3 allows the natural frequency of the structure to be expressed solely in terms of the acceleration due to gravity and the maximum static deflection, δ_s, that gravity produces. Of course, it is not necessary to limit the application of Eq. A-3 to systems which vibrate in the vertical plane. All that is required is an estimate of the maximum static deflection of the structure produced by a uniform 1-g acceleration field applied in the plane of vibration.

The accuracy of Eq. A-3 depends on the degree to which the static deformation under gravity conforms to the mode shape of vibration. Equation A-3 is exact for the simple spring-mass system shown in Fig. A-1 (a) and generally underestimates the natural frequency of more complex structures.

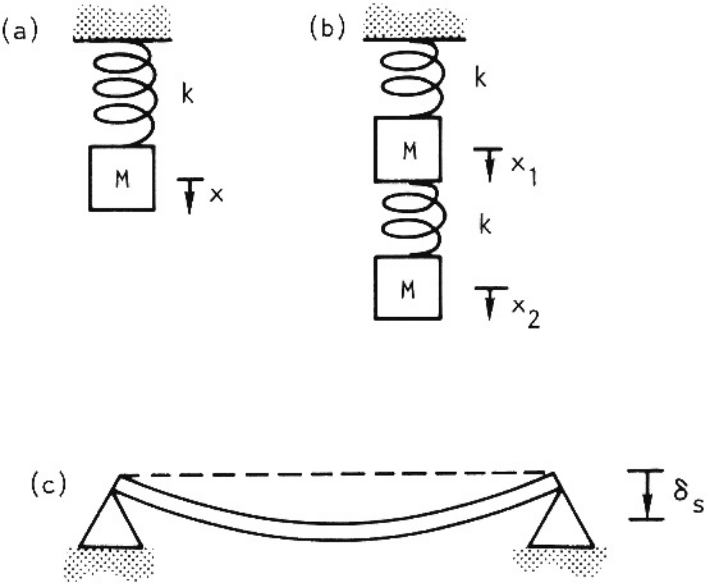

Fig. A-1. Elastic systems.

Two-Spring, Two-Mass System Example. Consider the system shown in Fig. A-1 (b). The maximum static deflection of this system under gravity is the deflection of the lower mass, which is easily computed to be:

$$\delta_s = \frac{2\ Mg}{k} + \frac{Mg}{k} = \frac{3\ Mg}{k}.$$

Using Eq. A-3, the fundamental natural frequency predicted from this deflection is

$$f = \frac{1}{2\pi}\frac{1}{3^{1/2}}\left(\frac{k}{M}\right)^{1/2}\ \text{Hz},$$

$$= \frac{0.5774}{2\pi}\left(\frac{k}{M}\right)^{1/2}\ \text{Hz},$$

and the mode shape, predicted from the static deflection, is

$$\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix} = \begin{pmatrix}1\\ 1.5\end{pmatrix}.$$

The exact solution for the fundamental natural frequency of this system is (frame 2 of Table 6-2):

$$f = \frac{0.6180}{2\pi}\left(\frac{k}{M}\right)^{1/2}\ \text{Hz},$$

and the exact mode shape in the fundamental mode is

$$\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix} = \begin{pmatrix}1\\ 1.618\end{pmatrix}.$$

Thus, Eq. A-3 underestimates the natural frequency of this system by about 6.6%. Equation A-3 cannot be used to estimate the natural frequency of the second, higher, mode of this system.

Beam Example. Consider the slender, uniform pinned-pinned beam shown in Fig. A-1(c). The static deflection of this beam at midspan due to gravity is (Ref. A-1):

$$\delta_s = \frac{5}{384} \frac{mgL^4}{EI} = \frac{1}{(2.960)^4} \frac{mgL^4}{EI},$$

where m is the mass per unit length of the beam, E is the modulus of elasticity, I is the area moment of inertia of the cross section, and L is the length of the beam. Using this result, the fundamental natural frequency of the beam can be estimated from Eq. A-3 to be:

$$f = \frac{(2.960)^2}{2\pi L^2} \left(\frac{EI}{m}\right)^{1/2} \text{ Hz.}$$

The exact result (Table 8-1) has the same form as this equation but with the factor 2.960 replaced by the factor π. Thus, the approximation of Eq. A-3 underestimates the fundamental natural frequency of the beam by about 11%.

Plate Example. It has been found that the maximum static deflection of a thin, uniform elliptical plate with a clamped edge due to its own weight is (Ref. A-2):

$$\delta_s = \frac{\mu g h a^4 b^4}{8(3a^4 + 2a^2b^2 + 3b^4)} \frac{12(1-\nu^2)}{Eh^3}.$$

h is the plate thickness, μ is the plate density, ν is Poisson's ratio, E is the modulus of elasticity, and a and b are the major and minor axes of the ellipse. Using this result, Eq. A-3 predicts that the fundamental natural frequency of the ellipse is:

$$f = \frac{2.828}{2\pi} \left(\frac{3a^4 + 2a^2b^2 + 3b^4}{\mu h a^4 b^4}\right)^{1/2} \left[\frac{Eh^3}{12(1-\nu^2)}\right]^{1/2} \text{ Hz.}$$

A more nearly exact analysis of the fundamental natural frequency of this plate gives (Ref. A-3):

$$f = \frac{3.612}{2\pi} \left(\frac{3a^4 + 2a^2b^2 + 3b^4}{\mu h a^4 b^4}\right)^{1/2} \left[\frac{Eh^3}{12(1-\nu^2)}\right]^{1/2} \text{ Hz.}$$

Thus, Eq. A-3 underestimates the fundamental natural frequency of the plate by about 22%.

The work of Jones (Ref. A-2) and Johns (Ref. A-4) suggests that the fundamental natural frequency of thin uniform plates can be accurately estimated by introducing a correction factor into Eq. A-3 to compensate for the underprediction:

$$f = \frac{1.277}{2\pi} \left(\frac{g}{\delta_s}\right)^{1/2} \text{ Hz.} \tag{A-4}$$

This equation has successfully predicted the fundamental natural frequencies of plates of a variety of shapes and boundary conditions to within 3% of the exact result (Ref. A-2).

REFERENCES

A-1. Roark, R., *Formulas for Stress and Strain*, 4th ed., McGraw-Hill, New York, 1965, p. 106.

A-2. Jones, R., "An Approximate Expression for the Fundamental Frequency of Vibration of Elastic Plates," *J. Sound Vib.* **38**, 503–504 (1975).

A-3. Mazumdar, J., "Transverse Vibration of Elastic Plates by the Method of Constant Deflection," *J. Sound Vib.* **18**, 147–155 (1971).

A-4. Johns, D. J., "Comments on 'An Approximate Expression for the Fundamental Frequency of Vibration of Elastic Plates'," *J. Sound Vib.* **41**, 385–387 (1975).